New Technique for Stability Analysis for Time-Varying Systems with Delay

Frédéric Mazenc Michael Malisoff

Abstract— **We propose a new stability analysis technique for a family of time-varying systems with delay. Under simple conditions, we prove exponential stability. The key ingredients in our proof are operators with integral terms, the notion of positive systems, and linear time-varying Lyapunov functionals. We illustrate our work using a chain of integrators.**

Key Words: **Delay, positive solutions, stability, time-varying.**

I. Introduction

The search for analysis and control methods for time-varying systems with delay is motivated by the fact that numerous physical phenomena and applications lead to nonlinear systems with delay [1], [2], [7], [17], which then lead to time-varying systems when a trajectory must be tracked. Analyzing these systems can be difficult, because many classical techniques do not apply. Notably, the usual frequency domain approach, and the construction of Lyapunov functions through linear matrix inequality techniques, do not apply to time-varying systems with delays. Therefore, despite their importance, not many contributions are devoted to time-varying systems with delays.

Moreover, most of the existing results are concerned with control problems for systems where the delay only occurs in the input. This is the case for [18] and [19], where the design of control laws for time-varying systems with input delays is done using the reduction model approach. Prediction is an important approach for systems with input delays that makes it possible to compensate arbitrarily long input delays; see, e.g., the significant work [1], [2], [10] of Bekiaris-Liberis, Karafyllis, Krstic, and others. However, the delay compensation approach may not always apply when the delays are in the vector fields of the system, which is the case we study in this note. Also, it may not always be easy to find the types of Lyapunov functions that can be transformed into the Lyapunov-Krasovskii functionals that are usually used to prove stability properties for delay systems.

Recently, a different type of result was obtained in [14], which establishes delay independent stability results for systems of neutral type. The approach in [14] uses the notions of nonnegative and cooperative systems, and relies on linear Lyapunov functions. In the present work, we take advantage of similar tools to develop a new stability analysis technique for linear time-varying systems with a pointwise delay, which are not necessarily periodic in time. The approach to proving stability that we propose here uses three key ingredients. First, we introduce operators with integral terms that lead to the study of a system with a distributed delay, coupled with an integral equation. Next, we prove that all of the solutions of this system are components of the solutions of a nonnegative system of higher dimension. Finally, we construct a linear time-varying Lyapunov-Krasovskii functional for the higher dimensional system. The functional is nonnegative on the positive orthant. Moreover, along all trajectories contained in the positive orthant, the time derivatives of the Lyapunov functionals are negative definite, and this leads to the desired Lyapunov decay conditions along all solutions. The Lyapunov functional construction we present here owes a great deal to the time-invariant ones in [9] for time-invariant systems with a pointwise delay; see also [5] and [8].

We use decompositions of certain functions that involve their cooperative parts, as was done, e.g., in [4], [5] and [15], in the contexts of time invariant systems or interval observers. This technique shares similarities with the internal positive representation that is presented in [6] and developed in subsequent papers (such as [3]). The two main advantages of the technique we propose here are that (a) its assumptions are relatively simple and that (b) it makes it possible to establish exponential stability for systems for which no other known techniques seem to apply. The stability results are established under an assumption on the size of the delay, but the actual value of the delay does not need to be known.

The rest of this note is organized as follows. Section II provides the required definitions. We state and prove our main result in Sections III-IV. In Section V, we provide illustrations, including an application to a key chain of integrators system. In Section VI, we summarize the value added by our work, and we suggest future research directions.

II. Definitions and Notation

In what follows, the dimensions of all vectors and matrices are arbitrary positive integers. For any matrix $M \in \mathbb{R}^{p\times q}$, we let $m_{i,j}$ denote its entry in row i and column j for all i and j. The $k\times n$ matrix in which each entry is 0 is also denoted by 0. The usual Euclidean norm $||x||_2 = \sqrt{x_1^2 + x_2^2 + \ldots + x_n^2}$ of vectors, and the induced norm of matrices of any dimensions, are denoted by $|\cdot|$. All inequalities must be understood componentwise, i.e., given vectors $v_a = (v_{a1}, ..., v_{ar})^\top \in \mathbb{R}^r$ and $v_b = (v_{b1}, ..., v_{br})^\top \in \mathbb{R}^r$, we write $v_a \leq v_b$ to mean that for all $i \in \{1, ..., r\}$, we have $v_{ai} \leq v_{bi}$. A square matrix is said to be *cooperative* or *Metzler* provided all of its off-diagonal entries are nonnegative. A matrix $M \in \mathbb{R}^{r\times s}$ is said to be *nonnegative* (resp., *positive*) provided every entry

F. Mazenc is with EPI DISCO INRIA-Saclay, L2S, CNRS Supelec, 3 Joliot Curie, 91192 Gif-sur-Yvette, France. `Frederic.Mazenc@lss.supelec.fr`.

M. Malisoff is with the Department of Mathematics, 303 Lockett Hall, Louisiana State University, Baton Rouge, LA 70803-4918, USA. `malisoff@lsu.edu`. Malisoff was supported by NSF-ECCS Grant 1102348, and by Roy P. Daniels Professorship #3 in the Louisiana State University College of Science.

$m_{i,j}$ of M satisfies $m_{i,j} \geq 0$ (resp., > 0). For simplicity, we always take the initial times for the trajectories of our systems to be $t_0 = 0$.

For any matrix $M = [m_{i,j}]$, let M^+ be the matrix whose position (i,j) entry is $\max\{0, m_{i,j}\}$ for all i and j, $M^- = M^+ - M$, $\overline{M}$ be the matrix whose diagonal entries are $m_{i,i}$ and whose off-diagonal entries are $\max\{0, m_{i,j}\}$, and $\underline{M} = \overline{M} - M$. Let $M^s = M^+ + M^-$, so M^s is obtained by taking the absolute values of all entries of M, and $M^* = \overline{M} + \underline{M}$. Let C^1 denote the set of all continuously differentiable functions, whose domains and ranges will be clear from the context. Given any constant $\tau > 0$, we let $C([-\tau, 0], \mathbb{R}^n)$ denote the set of all continuous $\mathbb{R}^n$-valued functions defined on a given interval $[-\tau, 0]$. We often abbreviate this set as C_{in}, and we call it the set of all *initial functions*. A system is *positive* for a class of initial functions $\mathcal{C}_0$ provided for each positive valued initial function in $\mathcal{C}_0$, the unique solution stays positive for all $t \geq 0$. For any continuous function $\varphi : [-\tau, +\infty) \to \mathbb{R}^n$ and all $t \geq 0$, we define φ_t by $\varphi_t(m) = \varphi(t+m)$ for all $m \in [-\tau, 0]$, i.e., the translation operator.

III. Statement of Result and Discussion

We first provide a stability analysis for linear time-varying systems of the form

$$\dot{x}(t) = A_1(t)x(t) + A_2(t)x(t-\tau) \tag{1}$$

where x is valued in $\mathbb{R}^n$ for any dimension n, $\tau > 0$ is the constant delay, the initial functions are in C_{in}, and $A_1 : [0, +\infty) \to \mathbb{R}^{n\times n}$ and $A_2 : [0, +\infty) \to \mathbb{R}^{n\times n}$ are continuous functions (but see Section V for an extension, based on combining our results for (1) with linearizations and backstepping). This includes the important case of time-varying linear systems with linear feedbacks with input delays, and linearizations of nonlinear systems whose linearizations have such a form.

To simplify the statements of our results, we define

$$B_1(t) = A_1(t) + A_2(t+\tau), \quad B_2(t) = -A_2(t+\tau), \tag{2}$$

and

$$B_3(t,m) = B_1(t)B_2(m) \tag{3}$$

and we introduce two assumptions:

Assumption 1: The matrix A_1 is bounded, and there are constants $\mathfrak{c}_j > 0$ and a C^1 function $p : [0, \infty) \to \mathbb{R}^n$ such that

$$\begin{array}{rr} \dot{p}(t)^\top + p(t)^\top \left(\bar{B}_1(t) + \underline{B}_1(t)\right) \leq -\mathfrak{c}_1 p(t)^\top, & (4) \\ \mathfrak{c}_2(1 \,...\, 1)^\top \leq p(t) \leq \mathfrak{c}_3(1 \,...\, 1)^\top, & (5) \\ p^\top(t)B_2^s(t) \leq \mathfrak{c}_4 p^\top(t), \text{ and} & \\ p^\top(t)B_3^s(t,m) \leq \mathfrak{c}_6 p^\top(t) & \end{array}$$

,

for all $t \geq 0$ and $m \geq 0$. □

Setting $\mathfrak{c}_5 = \mathfrak{c}_3\mathfrak{c}_4/\mathfrak{c}_2$, we also assume:

Assumption 2: The delay τ satisfies

$$\left(\frac{\mathfrak{c}_6}{\mathfrak{c}_1} + \mathfrak{c}_5\right)\tau < 1, \tag{6}$$

where the $\mathfrak{c}_i$'s are from Assumption 1. □

We can then prove:

Theorem 1: If (1) satisfies Assumptions 1-2, then (1) is uniformly globally exponentially stable to 0 for all initial functions. □

Remark 1: The matrices M^+, M^-, $\underline{M}$ and M^s are nonnegative, for all choices of M. However, $\overline{M}$ and M^* are not necessarily nonnegative, because their diagonal entries are the same as the corresponding diagonal entries of M. □

Remark 2: By replacing B_2^s and B_3^s by zero matrices in Step 3 of our proof of Theorem 1, we deduce that the system $\dot{X} = B_1^*(t)X$ is exponentially stable. □

Remark 3: For special cases where B_1^* is constant and Hurwitz, there is a vector $p_0 > 0$ such that (4) holds with $p(t) = p_0$ for all $t \geq 0$ [9]. We can sometimes use a time-varying change of coordinates to transform $\dot{X} = B_1(t)X$ into an autonomous system [11]. This may facilitate constructing a function p that satisfies Assumption 1. □

Remark 4: Assumption 2 restricts the size of τ, but it does not require that τ be known. Instead, only an upper bound for τ is required. Assumptions 1-2 also ensure that A_1 and A_2 are bounded. One cannot expect a stability result without assuming that τ is sufficiently small, because Assumptions 1-2 do not imply that the system $\dot{X} = A_1(t)X$ is asymptotically stable. See Section V for an example satisfying our assumptions where $\dot{X} = A_1(t)X$ is not asymptotically stable. The key difference between [14] and Theorem 1 is that the potential stabilizing effect of the delayed term $A_2(t)x(t-\tau)$ is not taken into account in [14], but it is in Theorem 1, as we illustrate in Section V. □

Remark 5: Consider the special case where the system $\dot{X}(t) = B_1(t)X(t)$ admits a strict quadratic Lyapunov function and τ is sufficiently small. In such cases, we can use a classical construction of a strict quadratic Lyapunov function to prove that (9) is globally exponentially stable. These types of Lyapunov functionals can be deduced from [16]. However, finding a quadratic strict Lyapunov function for the system $\dot{X}(t) = B_1(t)X(t)$ may be difficult since $B_1(t)$ is time-varying and not necessarily periodic, and may be zero for some t. The proof of Theorem 1 uses linear Lyapunov functionals whose expression is explicit when the function p in Assumption 1 is explicitly known. □

IV. Proof of Theorem 1

Step 1: Obtaining an Equivalent System

Since (1) is linear, all of its trajectories are defined over $[-\tau, +\infty)$. We introduce the time-varying operator

$$\xi(t) = x(t) + \int_{t-\tau}^{t} A_2(m+\tau)x(m)\mathrm{d}m\,. \tag{7}$$

Its time derivative along all trajectories of (1) satisfies

$$\begin{array}{rcl} \dot{\xi}(t) & = & [A_1(t) + A_2(t+\tau)]x(t) \\ & = & B_1(t)\left[\xi(t) + \int_{t-\tau}^{t} B_2(m)x(m)\mathrm{d}m\right] \end{array} \tag{8}$$

for all $t \geq 0$. We deduce that

$$\left\{ \begin{array}{rcl} \dot{\xi}(t) & = & B_1(t)\xi(t) + \int_{t-\tau}^{t} B_3(t,m)x(m)\mathrm{d}m \\ x(t) & = & \xi(t) + \int_{t-\tau}^{t} B_2(m)x(m)\mathrm{d}m\,. \end{array} \right. \tag{9}$$

It follows that all of the solutions of (1) converge to the origin if all the solutions of (9) with initial conditions in $(\phi_\xi, \phi_x) \in C_{\rm in}$ satisfying the matching condition

$$\phi_x(0) = \phi_\xi(0) + \int_{-\tau}^{0} B_2(m)\phi_x(m)\mathrm{d}m \tag{10}$$

are defined over $[0, +\infty)$ and converge exponentially to 0.

In fact, one can prove that all the solutions of (9) with initial conditions $(\phi_\xi, \phi_x) \in C_{\rm in}$ satisfying the matching condition (10) are continuous, uniquely defined, and defined over $[-\tau, +\infty)$ by noticing that they satisfy

$$\left\{\begin{array}{rcl} \dot{\xi}(t) & = & B_1(t)\xi(t) + \int_{t-\tau}^{t} B_3(t,m)x(m)\mathrm{d}m \\ \dot{x}(t) & = & B_1(t)\xi(t) + \int_{t-\tau}^{t} B_3(t,m)x(m)\mathrm{d}m \\ & & + B_2(t)x(t) - B_2(t-\tau)x(t-\tau) \end{array}\right. \tag{11}$$

for all $t \geq 0$, which is a classical system with distributed and pointwise delay for which the solutions are unique and defined over $[-\tau, +\infty)$. The time-varying system (11) is an interconnection with a distributed delay. To analyze the stability of (11), we write $B_1(t) = \overline{B}_1(t) - \underline{B}_1(t)$, $B_2(m) = B_2^+(m) - B_2^-(m)$, and $B_3(t,m) = B_3^+(t,m) - B_3^-(t,m)$, and observe that (9) is equivalent to

$$\left\{\begin{array}{rcl} \dot{\xi}(t) & = & \overline{B}_1(t)\xi(t) - \underline{B}_1(t)\xi(t) \\ & & + \int_{t-\tau}^{t} B_3^+(t,m)x(m)\mathrm{d}m \\ & & - \int_{t-\tau}^{t} B_3^-(t,m)x(m)\mathrm{d}m \\ x(t) & = & \xi(t) + \int_{t-\tau}^{t} B_2^+(m)x(m)\mathrm{d}m \\ & & - \int_{t-\tau}^{t} B_2^-(m)x(m)\mathrm{d}m. \end{array}\right. \tag{12}$$

In the next step, we analyze (12) by embedding its trajectories into those of a higher dimensional system.

Step 2: Analyzing the Equivalent System (12)

To analyze the stability properties of (12), we introduce the variables $Z(t) = -x(t)$ and $\Psi(t) = -\xi(t)$. We deduce from (12) that for all $t \geq 0$, the equalities

$$\left\{\begin{array}{rcl} \dot{\xi}(t) & = & \overline{B}_1(t)\xi(t) + \underline{B}_1(t)\Psi(t) \\ & & + \int_{t-\tau}^{t} B_3^+(t,m)x(m)\mathrm{d}m \\ & & + \int_{t-\tau}^{t} B_3^-(t,m)Z(m)\mathrm{d}m \\ x(t) & = & \xi(t) + \int_{t-\tau}^{t} B_2^+(m)x(m)\mathrm{d}m \\ & & + \int_{t-\tau}^{t} B_2^-(m)Z(m)\mathrm{d}m \\ \dot{\Psi}(t) & = & \overline{B}_1(t)\Psi(t) + \underline{B}_1(t)\xi(t) \\ & & + \int_{t-\tau}^{t} B_3^+(t,m)Z(m)\mathrm{d}m \\ & & + \int_{t-\tau}^{t} B_3^-(t,m)x(m)\mathrm{d}m \\ Z(t) & = & \Psi(t) + \int_{t-\tau}^{t} B_2^+(m)Z(m)\mathrm{d}m \\ & & + \int_{t-\tau}^{t} B_2^-(m)x(m)\mathrm{d}m \end{array}\right. \tag{13}$$

are satisfied. Thus, $(\xi, x, -\xi, -x)$ is a solution of (13), if (ξ, x) is a solution of (12). It follows that if all the solutions of (13) satisfying the matching condition

$$\begin{array}{rcl} x(0) & = & \xi(0) + \int_{-\tau}^{0} B_2^+(m)x(m)\mathrm{d}m \\ & & + \int_{-\tau}^{0} B_2^-(m)Z(m)\mathrm{d}m \\ Z(0) & = & \Psi(0) + \int_{-\tau}^{0} B_2^+(m)Z(m)\mathrm{d}m \\ & & + \int_{-\tau}^{0} B_2^-(m)x(m)\mathrm{d}m \end{array} \tag{14}$$

are continuous on $[-\tau, +\infty)$ and converge exponentially to 0, then all the solutions of (9) converge exponentially to 0.

Arguing as we did when we studied the existence of the solutions of (9), one can prove that all solutions of (13) satisfying (14) are continuous and defined over $[-\tau, +\infty)$. Next, we analyze the stability properties of (13). We prove in the appendix that (13) is positive for the class $\mathcal{C}_0$ of all initial functions satisfying the matching condition (14). Moreover, it is linear. Hence, it is globally exponentially stable if it is globally exponentially stable on only the positive orthant. To see why, let $\mathcal{X}$ be any solution of (13) with any initial condition $\phi_{\mathcal{X}} = (\phi_\xi, \phi_x, \phi_\Psi, \phi_Z)$ satisfying (14). Then we can find a positive valued solution $\mathcal{X}_a$ of (13) satisfying the matching condition, and a negative valued solution $\mathcal{X}_b$ of (13) satisfying the matching condition, such that the corresponding initial functions $\phi_{\mathcal{X}_a}$ and $\phi_{\mathcal{X}_b}$ satisfy

$$\phi_{\mathcal{X}_b}(t) < \phi_{\mathcal{X}}(t) < \phi_{\mathcal{X}_a}(t) \ \text{ for all } t \in [-\tau, 0]. \tag{15}$$

To see why such $\mathcal{X}_a$ and $\mathcal{X}_b$ exist, it suffices to find a negative valued function $\phi_{\mathcal{X}_b} : [-\tau, 0] \to \mathbb{R}^{4n}$ and a positive valued function $\phi_{\mathcal{X}_a} : [-\tau, 0] \to \mathbb{R}^{4n}$ satisfying (15), and both satisfying the matching conditions, since then the positivity of $\mathcal{X}_a$ and the negativity of $\mathcal{X}_b$ follow from our proof of the positivity of (13) in the appendix. To find $\phi_{\mathcal{X}_b}$ and $\phi_{\mathcal{X}_a}$ satisfying these requirements, let $L_{i,j}$ be the corresponding entries of B_2^s, so $B_2^s(m) = [L_{i,j}(m)]$ for all m. Since B_2^s is bounded, we can find a constant $H > 0$ such that

$$\max_{1 \leq i \leq n} \int_{-\tau}^{0} \sum_{j=1}^{n} L_{i,j}(m) e^{Hm} \mathrm{d}m \leq 0.5. \tag{16}$$

Let δ_i be the integral in (16), so $\delta_i \in [0, 1)$ for all $i \in \{1, 2, \ldots, n\}$. Set $E_*(t) = (e^{Ht}, e^{Ht}, \ldots, e^{Ht})^\top \in \mathbb{R}^n$ and $\Delta_* = (1-\delta_1, 1-\delta_2, \ldots, 1-\delta_n)^\top \in \mathbb{R}^n$. Then the positive valued function $\phi_d(t) = (\Delta_*, E_*(t), \Delta_*, E_*(t))^\top$ satisfies (14). Set $\bar{\delta} = \max_i \delta_i$, $\underline{\phi} = \min\{1 - \bar{\delta}, e^{-H\tau}\}$, and $\bar{\phi} = 1 + \max\{|\phi_{\mathcal{X}}(t)| : t \in [-\tau, 0]\}$. Then $\phi_{\mathcal{X}_a}(t) = (\bar{\phi}/\underline{\phi})\phi_d(t)$ is positive valued and $\phi_{\mathcal{X}_b}(t) = -(\bar{\phi}/\underline{\phi})\phi_d(t)$ is negative valued, and these choices satisfy (15).

Next, assume that (13) satisfies the required exponential stability property on the positive orthant. Then, since positivity of the system gives positivity of $\mathcal{X}_a(t) - \mathcal{X}(t)$ and $\mathcal{X}(t) - \mathcal{X}_b(t)$ for all $t \geq 0$, we get $\lim_{t \to +\infty}(\mathcal{X}_a(t) - \mathcal{X}(t)) = \lim_{t \to +\infty}(\mathcal{X}(t) - \mathcal{X}_b(t)) = 0$, so $\lim_{t \to +\infty}(\mathcal{X}_a(t) - \mathcal{X}_b(t)) = 0$, where the limits are exponential convergence. Since positivity of (13) gives $\mathcal{X}_b(t) \leq \mathcal{X}(t) \leq \mathcal{X}_a(t)$ for all $t \geq 0$, it follows that $\mathcal{X} = (\mathcal{X} - \mathcal{X}_b) + (\mathcal{X}_b - \mathcal{X}_a) + \mathcal{X}_a \to 0$ exponentially. Hence, our next step studies positive valued solutions of (13) with initial conditions satisfying (14).

Step 3: Exponential Stability of Positive Solutions of (13)

Set $c(t) = x(t) + Z(t)$ and $\gamma(t) = \xi(t) + \Psi(t)$. Then (13) and the decompositions $B_i^s = B_i^+ + B_i^-$ for $i = 2, 3$ and $B_1^* = \bar{B}_1 + \underline{B}_1$ give

$$\left\{\begin{array}{rcl} \dot{\gamma}(t) & = & B_1^*(t)\gamma(t) + \int_{t-\tau}^{t} B_3^s(t,m)c(m)\mathrm{d}m \\ c(t) & = & \gamma(t) + \int_{t-\tau}^{t} B_2^s(m)c(m)\mathrm{d}m. \end{array}\right. \tag{17}$$

We use the linear function $V_1 : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ defined by

$$V_1(t, \gamma) = p(t)^\top \gamma \ , \tag{18}$$

where p is from Assumption 1. Its time derivative along all trajectories of (17) satisfies

$$\begin{array}{rcl} \dot{V}_1(t) & = & p(t)^\top B_1^*(t)\gamma(t) + \dot{p}(t)^\top \gamma(t) \\ & & + p(t)^\top \int_{t-\tau}^t B_3^s(t,m)c(m)\mathrm{d}m \\ & = & [\dot{p}(t)^\top + p(t)^\top B_1^*(t)]\,\gamma(t) \\ & & + \int_{t-\tau}^t p(t)^\top B_3^s(t,m)c(m)\mathrm{d}m \end{array} \tag{19}$$

for all $t \geq 0$, where $\dot{V}_1(t) = \frac{d}{dt}(V_1(t, \gamma(t)))$. Using Assumptions 1 and 2 and setting $v(t) = V_1(t, \gamma(t))$, it follows that

$$\begin{array}{rcl} \dot{v}(t) & \leq & -\mathfrak{c}_1 v(t) + \mathfrak{c}_6 \int_{t-\tau}^t p(t)^\top c(m)\mathrm{d}m \\ p(t)^\top c(t) & \leq & v(t) + \mathfrak{c}_5 \int_{t-\tau}^t p^\top(m)c(m)\mathrm{d}m, \end{array} \tag{20}$$

hold for all $t \geq 0$, where we combined (17) and (19) and used the positivity of the solution. We next prove:

Claim 1: There are constants $g \in (\tau, 1/\mathfrak{c}_5)$ and $h > 0$ such that

$$hg - \mathfrak{c}_1 < 0 \ \text{ and } \ \mathfrak{c}_6 + h(g\mathfrak{c}_5 - 1) < 0 \tag{21}$$

hold, where $\mathfrak{c}_5$ and $\mathfrak{c}_6$ are from Assumption 2. □

Proof of Claim 1. From (6) in Assumption 2, we get $\tau < 1/\mathfrak{c}_5$, and we can find a constant $g \in (\tau, 1/\mathfrak{c}_5)$ such that $1 > (\mathfrak{c}_6/\mathfrak{c}_1 + \mathfrak{c}_5)g$. On the other hand, we can rewrite our goal (21) as

$$\frac{\mathfrak{c}_6}{1 - g\mathfrak{c}_5} < h < \frac{\mathfrak{c}_1}{g}. \tag{22}$$

There exists $h > 0$ such that the previous inequalities are satisfied if and only if there is $g \in (\tau, 1/\mathfrak{c}_5)$ such that $(1 - g\mathfrak{c}_5)/\mathfrak{c}_6 > g/\mathfrak{c}_1$. This inequality is equivalent to $1 > (\mathfrak{c}_6/\mathfrak{c}_1 + \mathfrak{c}_5)g$, which proves the claim. □

We use g from Claim 1 to define V_2 by

$$V_2(t, c_t) = \int_{t-\tau}^t (g - t + \ell) p^\top(\ell)c(\ell)\mathrm{d}\ell \ . \tag{23}$$

By the second inequality in (20), the time derivative of $V_2(t, c_t)$ along all positive solutions of (17) satisfies

$$\begin{array}{rcl} \dot{V}_2(t) & = & -\int_{t-\tau}^t p^\top(\ell)c(\ell)\mathrm{d}\ell + gp(t)^\top c(t) \\ & & -(g-\tau)p(t-\tau)^\top c(t-\tau) \\ & \leq & -\int_{t-\tau}^t p^\top(\ell)c(\ell)\mathrm{d}\ell + gv(t) \\ & & + g\mathfrak{c}_5 \int_{t-\tau}^t p^\top(m)c(m)\mathrm{d}m \\ & = & (g\mathfrak{c}_5 - 1)\int_{t-\tau}^t p^\top(\ell)c(\ell)\mathrm{d}\ell + gv(t) \end{array} \tag{24}$$

for all $t \geq 0$, since the fact that $g \geq \tau$ allows us to drop the term $(g-\tau)p(t-\tau)^\top c(t-\tau)$. Along all positive valued solutions of (13),

$$\begin{array}{rcl} V_2(t, c_t) & \geq & (g-\tau)\int_{t-\tau}^t p^\top(\ell)c(\ell)\mathrm{d}\ell \\ & \geq & (g-\tau)\mathfrak{c}_2 \displaystyle\int_{t-\tau}^t \sum_{i=1}^n c_i(\ell)\mathrm{d}\ell \end{array} \tag{25}$$

for all $t \geq 0$, by our lower bound on $p(t)$ from (5).

We define V_3 by

$$V_3(t, \gamma(t), c_t) = V_1(t, \gamma(t)) + hV_2(t, c_t) \ , \tag{26}$$

where $h > 0$ is from Claim 1 and V_1 is from (18). Along positive valued solutions of (17), we use (20) and (24) to get

$$\begin{array}{rcl} \dot{V}_3(t) & \leq & -\mathfrak{c}_1 v(t) + \mathfrak{c}_6 \int_{t-\tau}^t p^\top(m)c(m)\mathrm{d}m \\ & & + h(g\mathfrak{c}_5 - 1)\int_{t-\tau}^t p^\top(\ell)c(\ell)\mathrm{d}\ell + hgv(t) \\ & \leq & (hg - \mathfrak{c}_1)\, v(t) \\ & & + \left[\mathfrak{c}_6 + h(g\mathfrak{c}_5 - 1)\right]\int_{t-\tau}^t p^\top(m)c(m)\mathrm{d}m \end{array} \tag{27}$$

for all $t \geq 0$. Then, since $V_2(t, c_t) \leq g\int_{t-\tau}^t p^\top(\ell)c(\ell)\mathrm{d}\ell$ for all t, we can use (21) to find a constant $\mathfrak{c}_7 > 0$ such that $\dot{V}_3(t) \leq -\mathfrak{c}_7 V_3(t, \gamma(t), c_t)$ for all $t \geq 0$, namely, $\mathfrak{c}_7 = \min\{\mathfrak{c}_1 - hg, (h(1 - g\mathfrak{c}_5) - \mathfrak{c}_6)/(gh)\}$. Therefore, for all $t_1 \geq t_2 \geq 0$, we can integrate the preceding inequality to get

$$\begin{array}{l} V_1(t_1, \gamma(t_1)) + hV_2(t_1, c_{t_1}) \\ \leq e^{-\mathfrak{c}_7(t_1 - t_2)}\left[V_1(t_2, \gamma(t_2)) + hV_2(t_2, c_{t_2})\right] \ , \end{array} \tag{28}$$

by our formula (26) for V_3. Since (25) holds for all $t \geq 0$ along all positive trajectories of (13), Assumption 1 gives

$$\begin{array}{l} \mathfrak{c}_2 \displaystyle\sum_{i=1}^n \gamma_i(t_1) + h(g-\tau)\mathfrak{c}_2 \sum_{i=1}^n \int_{t_1-\tau}^{t_1} c_i(\ell)\mathrm{d}\ell \\ \leq e^{-\mathfrak{c}_7(t_1-t_2)}\big[V_1(t_2, \gamma(t_2)) + hV_2(t_2, c_{t_2})\big]. \end{array} \tag{29}$$

Since $B_2^s(m)$ is bounded in norm, it follows from the previous inequality that

$$\gamma(t) \to 0 \ \text{ and } \int_{t-\tau}^t B_2^s(m)c(m)\mathrm{d}m \to 0 \tag{30}$$

exponentially. From (17), we deduce that $c(t)$ converges exponentially to zero. Since $(\xi(t), x(t), \Psi(t), Z(t)) > 0$ for all $t \geq 0$, and since $c(t) = x(t) + Z(t)$ and $\gamma(t) = \xi(t) + \Psi(t)$ hold, it follows that $\xi(t)$, $x(t)$, $\Psi(t)$, and $Z(t)$ all converge exponentially to zero. Therefore, all of the positive valued solutions of (13) converge exponentially to zero. Hence, our argument at the end of Step 2 implies that (13) is globally exponentially stable. It follows that (9) is globally exponentially stable as well. This proves our theorem.

V. Illustrations

A. System with a stabilizing term without delay

We first consider the one-dimensional system

$$\dot{x}(t) = l_1 \cos^2(t)x(t) + l_2 \sin(t)x(t-\tau) \ , \tag{31}$$

where $\tau \geq 0$, $l_1 \in \mathbb{R}$, and $l_2 \in \mathbb{R}$ are constants. We use Theorem 1 to find conditions on τ, l_1 and l_2 that ensure that (31) is exponentially stable. Using the above notation, we have $B_1(t) = l_1\cos^2(t) + l_2\sin(t+\tau)$, $B_2(t) = -l_2\sin(t+\tau)$, $B_1^*(t) = B_1(t)$, and $B_3(t,m) = -l_2\sin(m+\tau)(l_1\cos^2(t) + l_2\sin(t+\tau))$.

Let us determine conditions ensuring that there are a positive function $p : \mathbb{R} \to \mathbb{R}$ of class C^1 and a constant $\mathfrak{c}_1 > 0$ such that

$$\dot{p}(t) + p(t)B_1(t) = -\mathfrak{c}_1 p(t) \tag{32}$$

for all $t \geq 0$. Equality (32) is equivalent to

$$\dot{p}(t) = -[\mathfrak{c}_1 + l_1\cos^2(t) + l_2\sin(t+\tau)]p(t) \ , \tag{33}$$

Assuming that $l_1 < 0$ and choosing $\mathfrak{c}_1 = -\frac{l_1}{2}$, we obtain

$$\dot{p}(t) = \left[-\tfrac{l_1}{2}\cos(2t) - l_2\sin(t+\tau)\right]p(t)\ . \tag{34}$$

Thus, we can choose the positive function

$$p(t) = e^{-\frac{l_1}{4}\sin(2t) + l_2\cos(t+\tau)}\ . \tag{35}$$

Therefore, Assumptions 1-2 hold with $\mathfrak{c}_2 = \exp(-|l_1|/4 - |l_2|)$, $\mathfrak{c}_3 = 1/\mathfrak{c}_2$, $\mathfrak{c}_5 = |l_2|\exp(|l_1|/2 + 2|l_2|)$, $\mathfrak{c}_6 = (|l_1| + |l_2|)|l_2|$, and all $\tau > 0$ such that $(\mathfrak{c}_6/\mathfrak{c}_1 + \mathfrak{c}_5)\,\tau < 1$, i.e.,

$$|l_2|\left(\tfrac{2}{|l_1|}(|l_1| + |l_2|) + e^{0.5|l_1| + 2|l_2|}\right)\tau \ < \ 1\ . \tag{36}$$

By Theorem 1, we conclude that the system (31) is globally exponentially stable, provided $l_1 < 0$ and (36) hold.

B. System with a stabilizing term with delay

We next illustrate Theorem 1 using a chain of integrators example, which will also show how we can use our theorem in conjunction with backstepping and linearization. We consider the system

$$\begin{array}{rcl} \dot{\xi}_1(t) & = & v_1(t-\tau) \\ \dot{\xi}_2(t) & = & v_2(t-\tau) \\ \dot{\xi}_3(t) & = & \xi_2(t)v_1(t), \end{array} \tag{37}$$

where we assume that the constant delay τ satisfies

$$\tau \in \left(0, \tfrac{1}{3+2\sqrt{e}}\right). \tag{38}$$

In [11, Section 6.2], we solved a tracking problem for (37) for the reference trajectory $(-\cos(t), 0, 0)^\top$ for the case where $\tau = 0$, by building a strict Lyapunov function for the tracking dynamics. However, there is no clear analog of this earlier construction under our condition (38) on the delay. Here we solve the problem of locally asymptotically tracking the trajectory $(\sin(t), 0, 0)^\top$ under the delay bound (38). To this end, we set $\gamma_1(t) = \xi_1(t) - \sin(t)$. Then we obtain

$$\begin{array}{rcl} \dot{\xi}_3(t) & = & \xi_2(t)v_1(t) \\ \dot{\xi}_2(t) & = & v_2(t-\tau) \\ \dot{\gamma}_1(t) & = & v_1(t-\tau) - \cos(t)\ . \end{array} \tag{39}$$

We set $v_1(t) = \cos(t+\tau) - \gamma_1(t)$ to obtain

$$\begin{array}{rcl} \dot{\xi}_3(t) & = & \xi_2(t)[\cos(t+\tau) - \gamma_1(t)] \\ \dot{\xi}_2(t) & = & v_2(t-\tau) \\ \dot{\gamma}_1(t) & = & -\gamma_1(t-\tau)\ . \end{array} \tag{40}$$

Then (38) and standard arguments imply that $\dot{\gamma}_1(t) = -\gamma_1(t-\tau)$ is globally exponentially stable to 0 (GES), by using the Lyapunov-Krasovskii functional $V(\gamma_t) = 0.5\gamma^2(t) + b\int_{t-2\tau}^{t}\int_s^t \gamma^2(r)\mathrm{d}r\mathrm{d}s$ for any constant $b \in (2/3, 3/4)$.

Next, note that the linear approximation of (40) at 0 is

$$\begin{array}{rcl} \dot{\xi}_3(t) & = & \xi_2(t)\cos(t+\tau) \\ \dot{\xi}_2(t) & = & v_2(t-\tau) \\ \dot{\gamma}_1(t) & = & -\gamma_1(t-\tau)\ . \end{array} \tag{41}$$

The origin of (41) is GES provided the origin of

$$\begin{array}{rcl} \dot{\xi}_3(t) & = & \cos(t+\tau)\xi_2(t) \\ \dot{\xi}_2(t) & = & v_2(t-\tau) \end{array} \tag{42}$$

is GES. To show the GES property for (42), we apply a backstepping approach. The time-varying change of variables

$$\gamma_2(t) = \xi_2(t) + \cos(t+\tau)\xi_3(t-\tau) \tag{43}$$

transforms (42) into

$$\begin{array}{rcl} \dot{\xi}_3(t) & = & -\cos^2(t+\tau)\xi_3(t-\tau) \\ & & +\cos(t+\tau)\gamma_2(t) \\ \dot{\gamma}_2(t) & = & v_2(t-\tau) - \sin(t+\tau)\xi_3(t-\tau) \\ & & +\cos(t+\tau)[-\cos^2(t)\xi_3(t-2\tau) \\ & & +\cos(t)\gamma_2(t-\tau)]\ . \end{array} \tag{44}$$

Choosing

$$\begin{array}{rcl} v_2(t-\tau) & = & \sin(t+\tau)\xi_3(t-\tau) - \gamma_2(t-\tau) \\ & & -\cos(t+\tau)[-\cos^2(t)\xi_3(t-2\tau) \\ & & +\cos(t)\gamma_2(t-\tau)] \end{array} \tag{45}$$

gives the triangular system

$$\begin{array}{l} \dot{x}_1(t) = -\cos^2(t+\tau)x_1(t-\tau) + \cos(t+\tau)x_2(t) \\ \dot{x}_2(t) = -x_2(t-\tau)\ , \end{array} \tag{46}$$

where $x_1 = \xi_3$ and $x_2 = \gamma_2$. Then Theorem 1 can be used to study the stability properties of the system (46). With the notation of the previous section, we have

$$\begin{array}{l} B_1(t) = \begin{bmatrix} -\cos^2(t+2\tau) & \cos(t+\tau) \\ 0 & -1 \end{bmatrix} \text{ and} \\ B_1^*(t) = \begin{bmatrix} -\cos^2(t+2\tau) & |\cos(t+\tau)| \\ 0 & -1 \end{bmatrix}. \end{array} \tag{47}$$

Let us consider the function $p : \mathbb{R} \to \mathbb{R}^2$ defined by

$$p(t) = \begin{bmatrix} e^{\frac{1}{4}\sin(2t+4\tau)} & q \end{bmatrix}^\top \tag{48}$$

where $q = 2e^{1/4}$. Then

$$\dot{p}(t) = \begin{bmatrix} \dfrac{1}{2}\cos(2t+4\tau)e^{\frac{1}{4}\sin(2t+4\tau)} & 0 \end{bmatrix}^\top\ . \tag{49}$$

We deduce that

$$\begin{array}{l} \dot{p}(t)^\top + p(t)^\top B_1^*(t) \\ = \begin{bmatrix} \left(\frac{1}{2}\cos(2t+4\tau) - \cos^2(t+2\tau)\right)e^{\frac{1}{4}\sin(2t+4\tau)} \\ |\cos(t+\tau)|e^{\frac{1}{4}\sin(2t+4\tau)} - q \end{bmatrix}^\top \\ = -\begin{bmatrix} \frac{1}{2}e^{\frac{1}{4}\sin(2t+4\tau)} \\ 2e^{\frac{1}{4}} - |\cos(t+\tau)|e^{\frac{1}{4}\sin(2t+4\tau)} \end{bmatrix}^\top \\ \leq -\begin{bmatrix} \frac{1}{2}e^{\frac{1}{4}\sin(2t+4\tau)} & e^{\frac{1}{4}} \end{bmatrix} \\ = -\frac{1}{2}p(t)^\top\ . \end{array}$$

Hence, Assumptions 1-2 hold with $\mathfrak{c}_1 = 0.5$, $\mathfrak{c}_2 = e^{-1/4}$, $\mathfrak{c}_3 = 2e^{1/4}$, $\mathfrak{c}_5 = 2\sqrt{e}$, and $\mathfrak{c}_6 = 3/2$, so Theorem 1 applies when (38) is satisfied. This allows us to conclude.

Remark 6: We are not aware of any other method to prove that the system (46) is globally exponentially stable when the bound (38) is satisfied. Since the preceding analysis uses a linearization, it only ensures local stability of the original system (37). However, we conjecture that the control laws we provide (or similar ones) actually globally asymptotically stabilize the reference trajectory. Establishing this result requires the study of the nonlinear system (40) and therefore it is beyond the scope of this note. □

VI. Conclusions

Stabilization of time-varying linear systems with delays is challenging and beyond the scope of the standard frequency domain and linear matrix inequality methods. While there is a large literature on stabilizing such systems under delays, the existing results are largely limited to systems with input delays, and they often require constructing complicated Lyapunov-Krasovskii functionals that may not always be easy to obtain [20]. Here we used a very different approach, based on expressing the original system as an interconnection of (a) an integral equation and (b) a nonlinear differential equation with a distributed delay. Another novel feature of our analysis is our viewing the system trajectories as being components of trajectories of a higher order system, and then using a positive system argument to reduce the stabilization problem to an analysis of positive valued solutions of the higher order system. This improved on the analogous results from the first author's work [14] for neutral systems, which did not take the potentially stabilizing effect of the delayed term into account. The positivity properties of the systems made it possible to use simple linear Lyapunov-Krasovskii functionals. We illustrated our work in a chain of integrators, which showed how our analysis gave a larger upper bound for the admissible delays than existing results. We plan to generalize our theorem to systems that are nonlinear in the state. We also hope to merge our results with the second author's works [12], [13] on robustness under perturbations and state constraints.

Appendix: Positiveness of the system (13)

Let $(\phi_\xi, \phi_x, \phi_\Psi, \phi_Z) \in C_{\rm in}$ be a positive initial condition satisfying the matching condition (14). Let us prove that the solution of (13) with $(\phi_\xi, \phi_x, \phi_\Psi, \phi_Z) \in C_{\rm in}$ as the initial function is positive for all $t \in [-\tau, +\infty)$. Throughout the sequel, we let $\overline{B}_{1,i,j}$ denote the (i,j) entry of $\overline{B}_1$ for all i and j. First, recall that the solution is continuous over $[-\tau, +\infty)$. Next, we prove this result by contradiction.

Case 1: Assume that there is $t_c > 0$ and $i \in \{1, ..., n\}$ such that $(\xi(t), x(t), \Psi(t), Z(t)) > 0$ for all $t \in [-\tau, t_c)$ and $\xi_i(t_c) = 0$. Since $\overline{B}_1(t)$ is Metzler and $\underline{B}_1(t)$, $B_3^+(t,m)$, and $B_3^-(t,m)$ are nonnegative for all $t \geq 0$ and all $m \geq 0$, it follows from the structure of the ξ subdynamics of (13) that for all $t \in [0, t_c]$, we have $\dot{\xi}_i(t) \geq \overline{B}_{1,i,i}(t)\xi_i(t)$. By integrating this inequality, we deduce that

$$\xi_i(t_c) \geq e^{\int_0^{t_c} \overline{B}_{1,i,i}(m)\mathrm{d}m}\xi_i(0) > 0. \tag{A.1}$$

This yields a contradiction.

Case 2: Assume that there is $t_c > 0$ and $i \in \{1, ..., n\}$ such that $(\xi(t), x(t), \Psi(t), Z(t)) > 0$ for all $t \in [-\tau, t_c)$ and $x_i(t_c) = 0$. Since $B_2^+(m)$ and $B_2^-(m)$ are nonnegative, it follows from the structure of the x subdynamics of (13) that $x_i(t_c) \geq \xi_i(t_c) \geq 0$. Consequently, $\xi_i(t_c) = 0$. From the first case, we are again led to a contradiction.

Case 3: Assume that there is $t_c > 0$ and $i \in \{1, ..., n\}$ such that $(\xi(t), x(t), \Psi(t), Z(t)) > 0$ for all $t \in [-\tau, t_c)$ and $\Psi_i(t_c) = 0$. Arguing as in the first case, we can conclude.

Case 4: Assume that there is $t_c > 0$ and $i \in \{1, ..., n\}$ such that $(\xi(t), x(t), \Psi(t), Z(t)) > 0$ for all $t \in [-\tau, t_c)$ and $Z_i(t_c) = 0$. Arguing as in the second case, we can conclude from Case 3. This concludes the proof.

References

[1] N. Bekiaris-Liberis and M. Krstic. *Nonlinear Control Under Nonconstant Delays*. Advances in Design and Control Series. Society for Industrial and Applied Mathematics, Philadelphia, PA, 2013.

[2] N. Bekiaris-Liberis and M. Krstic. Robustness of nonlinear predictor feedback laws to time- and state-dependent delay perturbations. *Automatica*, 49(6):1576–1590, 2013.

[3] F. Cacace, A. Germani, C. Manes, and R. Setola. A new approach to the internal positive representation of linear MIMO systems. *IEEE Trans. Automatic Control*, 57(1):119–134, 2012.

[4] D. Efimov, W. Perruquetti, T. Raissi, and A. Zolghadri. Interval observers for time-varying discrete-time systems. *IEEE Trans. Automatic Control*, 58(12):3218–3224, 2013.

[5] D. Efimov, W. Perruquetti, and J.-P. Richard. On reduced-order interval observers for time-delay systems. In *Proceedings of the European Control Conference*, pages 2116–2121, Zurich, Switzerland, 2013.

[6] A. Germani, C. Manes, and P. Palumbo. Representation of a class of MIMO systems via internally positive realization. *European J. Control*, 16(3):291–304, 2010.

[7] A. Germani, C. Manes, and P. Pepe. Input-output linearization with delay cancellation for nonlinear delay systems: the problem of the internal stability. *International J. Robust and Nonlinear Control*, 13(9):909–937, 2003.

[8] W. Haddad and V. Chellaboina. Stability and dissipativity theory for nonnegative and compartmental dynamical systems with time delay. *Systems and Control Letters*, 51(5):355–361, 2004.

[9] W. Haddad, V. Chellaboina, and Q. Hui. *Nonnegative and Compartmental Dynamical Systems*. Princeton Univ. Press, Princeton, NJ, 2010.

[10] I. Karafyllis and M. Krstic. Nonlinear stabilization under sampled and delayed measurements, and with inputs subject to delay and zero-order hold. *IEEE Trans. Automatic Control*, 57(5):1141–1154, 2012.

[11] M. Malisoff and F. Mazenc. *Constructions of Strict Lyapunov Functions*. Communications and Control Engineering Series. Springer, New York, 2009.

[12] M. Malisoff and F. Zhang. Stability and robustness analysis for curve tracking control using input-to-state stability. *IEEE Trans. Automatic Control*, 57(5):1320–1326, 2012.

[13] M. Malisoff and F. Zhang. Adaptive control for planar curve tracking under controller uncertainty. *Automatica*, 49(5):1411–1418, 2013.

[14] F. Mazenc. Stability analysis of time-varying neutral time-delay systems. *IEEE Trans. Automatic Control*, to appear.

[15] F. Mazenc, T. Dinh, and S.-I. Niculescu. Interval observers for discrete-time systems. *International J. Robust and Nonlinear Control*, to appear.

[16] F. Mazenc, H. Ito, and P. Pepe. Construction of Lyapunov functionals for coupled differential and continuous time difference equations. In *Proceedings of the IEEE Conference on Decision and Control*, pages 2245–2250, Florence, Italy, 2013.

[17] F. Mazenc and M. Malisoff. Asymptotic stabilization for feedforward systems with delayed feedback. *Automatica*, 49(3):780–787, 2013.

[18] F. Mazenc, M. Malisoff, and S.-I. Niculescu. Stabilization of linear time varying systems with input delays: Application to rapidly time varying systems. In *Proceedings of the IEEE Conference on Decision and Control*, pages 1584–1589, Florence, Italy, 2013.

[19] F. Mazenc and S.-I. Niculescu. Generating positive and stable solutions through delayed state feedback. *Automatica*, 47(3):525–533, 2011.

[20] P. Pepe, I. Karafyllis, and Z.-P. Jiang. On the Liapunov-Krasovskii methodology for the ISS of systems described by coupled delay differential and difference equation. *Automatica*, 49(9):2266–2273, 2008.